Statistical Analysis of On-Chip Power Delivery Networks Considering Lognormal Leakage Current Variations With Spatial Correlation

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***Abstract*—As the technology scales into 90 nm and below, process-induced variations become more pronounced. In this paper, we propose an efficient stochastic method for analyzing the voltage drop variations of on-chip power grid networks, considering log-normal leakage current variations with spatial correlation. The new analysis is based on the Hermite polynomial chaos (PC) representation of random processes. Different from the existing Hermite PC based method for power grid analysis (Ghanta *et al.*, 2005), which models all the random variations as Gaussian processes without considering spatial correlation, the new method consider both wire variations and subthreshold leakage current variations, which are modeled as log-normal distribution random variables, on the power grid voltage variations. To consider the spatial correlation, we apply orthogonal decomposition to map the correlated random variables into independent variables. Our experiment results show that the new method is more accurate than the Gaussian-only Hermite PC method using the Taylor expansion method for analyzing leakage current variations. It is two orders of magnitude faster than the Monte Carlo method with small variance errors. We also show that the spatial correlation may lead to large errors if not being considered in the statistical analysis.**

***Index Terms*—Hermite polynomials, leakage current, power delivery network, spectral analysis.**

I. Introduction

PROCESS-INDUCED variability has huge impacts on the circuit performance in the sub-90-nm VLSI technologies [17], [18]. One important aspect of the variations comes from the chip leakage currents. Leakage currents come from different sources. The dominant factor is the subthreshold leakage current. The reason is that subthreshold leakage current has a rapid increasing rate (about 5X-10X increase per technology generation [5]), and it is highly sensitive to threshold voltage $V_{\rm th}$ variations, owning to the exponential relationship between subthreshold current $I_{\rm off}$ and threshold voltage $V_{\rm th}$ as shown below [24]

$$I_{\rm off} = I_{s0} e^{\frac{V_{\rm gs}-V_{\rm th}}{nV_T}} \left(1 - e^{\frac{-V_{ds}}{V_T}}\right) \quad (1)$$

where I_{s0} is a constant related to the device characteristics, V_T is the thermal voltage, and n is a constant. It was shown in [13] that leakage variations for 90 nm can be 20×. Based on the ITRS 2005 [1], the leakage power accounts for more than 60% at 45 nm, there are many consequences for chip design, especially for design of the power grid. The grid will develop voltage drop at all the nodes that are correspondingly significant with a strong within-die components. The voltage drop is unavoidable and manifests itself as a background noise on the grid which has an impact on the circuit delay and operation.

Clearly, the leakage current has exponential dependency on the threshold voltage $V_{\rm th}$. In the sequel, the leakage current is mainly referred to as the subthreshold leakage current. Detailed analysis shows that $I_{\rm off}$ is also an exponential function of the effective channel length $L_{\rm eff}$ [20]. Actually $L_{\rm eff}$ are strongly correlated with $V_{\rm off}$ as $V_{\rm off}$ variations typically are caused by the $L_{\rm eff}$. So if we model $V_{\rm th}$ or $L_{\rm eff}$ as the random variables with Gaussian variation caused by the inter-die or intra-die process variations, then the leakage currents will have a log-normal distribution as shown in [20]. On top of this, those random variables are spatially correlated within a die, owning to the nature of the many physical and chemical manufacture processes [17].

On-chip power grid analysis and designs have been intensively studied in the past due to the increasing impacts of excessive voltage drops as technologies scale [2], [4], [11], [14], [21], [22], [25], [26], [29], [30], [32]. Owning to the increasing impacts of leakage currents and its variations on the circuit performances, especially on the on-chip power delivery networks, a number of research works have been proposed recently to perform the stochastic analysis of power grid networks under process-induced leakage current variations. The voltage drop of power grid networks subject to the leakage current variations was first studied in [6], [7]. This method assumes that the log-normal distribution of the node voltage drop caused by the log-normal leakage current inputs and is based on a localized Monte Carlo (MC) (sampling) method to compute the variance of the node voltage drop. However, this localized sampling method is limited to the static dc solution of power grids modeled as resistor-only networks. Therefore, it can only compute the responses to the standby leakage currents. However, the

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dynamic leakage currents become more significant, especially when the sleep transistors are intensively used nowadays for reducing leakage powers. In [19], [23], impulse responses are used to compute the means and variances of node voltage responses caused by general current variations. But this method needs to know the impulse response from all the current sources to all the nodes, which is expensive to compute for a large network. In [20], the probability density function (pdf) of leakage currents is computed based on the Gaussian variations of channel lengths.

Recently, a stochastic simulation method for interconnect and power grid networks has been proposed [10], [27], [28]. This method is based on the orthogonal polynomial chaos (PC) expansion of random processes to represent and solve for the stochastic responses of linear systems. The major benefit of this method is its compatibility with current transient simulation framework: it solves for some coefficients of the orthogonal polynomials, which can be done by using normal transient simulations of the original circuits with deterministic inputs to compute variances of node responses. Some existing approaches [10], [27] model all the parameter variations as Gaussian (or approximate them as Gaussian variations by using first-order Taylor expansion) [28]. Those methods also fail to consider the spatial correlation in the process parameter random variables.

In this paper, we apply the orthogonal polynomial based method (also called spectral statistical method) to deal with leakage current inputs with log-normal distributions and spatial correlations. We show how to represent a log-normal distribution in terms of Hermite polynomials, assuming Gaussian distribution of threshold voltage $V_{\rm th}$ in consideration of intra-die variation. To consider the spatial correlation, we apply orthogonal decomposition via principal component analysis to map the correlated random variables into independent variables. To the best knowledge of the authors, the proposed method is the first method being able to perform statistical analysis on power grids with variation dynamic leakage currents having lognormal distributions and spatial correlations. Experiment results show that the proposed method predicates the variances of the resulting log-normal-like node voltage drops more accurately than Taylor expansion based Gaussian approximation method.

We remark that we only consider the leakage current inputs with log-normal distributions to emphasize our new contributions. The reason is that leakage currents can be variable significantly. In 90 nm, it can lead to 20X variations [13]. For the coming 45 nm, it will dominate the currents of a chip (60% based on ITRS 2005 [1]). Considering variations from leakage currents has significant practice relevance. Also for general current variations from dynamic power of the circuits, which typically can be modeled as Gaussian distribution, existing work [10] using Taylor series expansion has been explored. The voltage variations caused by the dynamic power can be considered on top of the variations from the log-normal leakage currents. We notice that similar work, which consider only leakage variations have been done before [6], [7].

Also we also remark that Vdd drop will have impacts on the leakage currents, which create a negative feedback for the leakage current itself as increasing Vdd drop leads to lower $V_{\rm gs}$ in (1), which leads to smaller $I_{\rm off}$. However, to consider the effect, both the power grid and signal circuits need to be simulated together, which will be very expensive. Hence, practically, two-step simulation approach is used where power grid and signal circuits are simulated separately but in an iterative way to consider the coupling between them. In light of this simulation methodology, our proposed method can be viewed as the only one step (power grid simulation step) in such a method.

The rest of this paper is organized as follows. Section II presents models of power grid networks and the problem we try to solve. Section III reviews the orthogonal PC based stochastic simulation methods. Section IV presents our new orthogonal PC based method for stochastic analysis power grids subjecting to log-normal leakage current variations. Section V presents the method to deal with spatial correlation in the lognormal leakage current sources in this paper. Section VI considers the process variation influencing conductance, capacitors and leakage current sources. Section VII presents the experimental results and Section VIII concludes this paper.

II. Problem Formulation

In this section, we first present the model of power grids in this paper. We then present the modeling issue of leakage current under intra-die variations. After this, we present the problem that we try to solve.

A. Power Grid Network Models

The power grid networks in this paper are modeled as *RC* networks with known time-variant current sources, which are obtained by gate level logic simulations of the VLSI systems. For a power grid (versus the ground grid), some nodes have known voltage modeled as constant voltage sources. For C4 power grids, the known voltage nodes can be internal nodes inside the power grid. Given known deterministic vector of current sources, $I(t)$, the node voltages can be obtained by solving the following differential equations, which is formulated using modified nodal analysis (MNA) approach:

$$Gv(t) + C\frac{dv(t)}{dt} = I(t) \tag{2}$$

where G is the conductance matrix, C is admittance matrix resulting from capacitive elements. $v(t)$ is the vector of time-varying node voltages and branch currents of voltage sources that we try to solve.

B. Modeling Leakage Current Variations

The G and C matrices and input currents $I(t)$ depend on the circuit parameters, such as metal wire width, length, thickness on power grids, and transistor parameters, such as channel length, width, gate oxide thickness, etc. Some previous work assumes that all circuit parameters and current sources are treated as uncorrelated Gaussian random variables [10]. In this paper, we consider both power grid wire variations and the log-normal leakage current variations, caused by the channel length variations, which is modeled as Gaussian (normal) variations [20].

Process-induced variations can also be classified into inter-die (die-to-die) variations and intra-die variations. In inter-die variations, all the parameters variations are correlated.

The worst case corner can be easily found by setting the parameters to their range limits (mean plus 3σ). The difficulty lies in the intra-die variations, where the circuit parameters are not correlated or spatially correlated within a die. Intra-die variations also consist of local and layout dependent deterministic components and random components, which typically are modeled as multivariate Gaussian process with some spatial correlations [3]. In this paper, we first assume we have a number of independent (uncorrelated) transformed ortho-normal random Gaussian variables $\xi(\theta)$, $i = 1,\ldots,n$, which actually model the channel length and the device threshold voltage variations and other variations. Then, we consider spatial correlation in the intra-die variation. We apply the principal component analysis (PCA) method to transfer the correlated variables into un-correlated variables before the spectral statistical analysis.

Let Ω denote the sample space of the experimental or manufacturing outcomes. For $\omega \in \Omega$, let $\boldsymbol{\xi}_{\mathbf{d}}(\omega) = [\xi_{1d}(\omega),\ldots,\xi_{rd}(\omega)]$ be a vector of r Gaussian variables to represent the circuit parameters of interest. After the PCA operation, we obtain independent random variable vectors $\boldsymbol{\xi} = [\xi_1,\ldots,\xi_n]$. Notice that $n \leq r$ in general. Therefore, given the process variations, the MNA for (2) becomes

$$G(\boldsymbol{\xi})v(t) + C(\boldsymbol{\xi})\frac{dv(t)}{dt} = I\left(t,\boldsymbol{\xi}(\theta)\right). \tag{3}$$

The variation in wire width and thickness will cause variation in the conductance matrix $G(\xi)$ and capacitance matrix $C(\xi)$. The variations are more related to back-end-of-the-line (BEOL) as power grids are mainly metals at top or middle layers. The input current vector, $I(t,\boldsymbol{\xi}(\theta))$, has both deterministic and random components. In this paper, to simplify our analysis, we assume the dynamic currents (power) caused by circuit switching are still modeled as deterministic currents as we only consider the leakage variations. Practically, the variations caused by the dynamic power of circuits can be significant. But the voltage variations caused by the leakage variations can be viewed as background noise, which can be considered together with dynamic power-induced variations later.

To obtain the variation current sources $I(t,\boldsymbol{\xi}(\theta))$, some library characterization methods will be used to compute the $I(t,\boldsymbol{\xi}(\theta))$ once we know the effective channel length L_{eff} variations, threshold voltage (V_{th}) variations and other variable sources under different input patterns. With those variation-aware cell library, we can more accurately obtain the $I(t,\boldsymbol{\xi}(\theta))$ based on the logic simulation of the whole chip under some inputs.

Note that practical use perspective, user may be only interested in voltage variations over a period of time or worst case in a period of time. Those information can be easily obtained once we know the variations in any given time instance. In other words, the information we obtain here can be used to derive any other information that is interesting to designers.

The problem we need to solve is to efficiently find the mean and variances of voltage $v(t)$ at any node and at any time instance. A straightforward method is MC based sampling methods. We randomly generate $G(\boldsymbol{\xi})$, $C(\xi)$ and $I(t,\xi(\theta))$, which is based on the log-normal distribution, solve (3) in time domain for each sampling and compute the means and variances based on sufficient samplings. Obviously, MC will be computationally expensive. However, MC will give the most reliable results and is the most robust and flexible method.

III. Spectral Statistical Based Simulation

In this section, we briefly review the spectral statistical simulation method based on orthogonal PC representation of statistical processes.

A. Concept of Hermite PC

In the following, a random variable $\xi(\theta)$ is expressed as a function of θ, which is the random event. Hermite PC utilizes a series of orthogonal polynomials (with respect to the Gaussian distribution) to facilitate stochastic analysis [31]. These polynomials are used as the orthogonal base to decompose a random process in a similar way that sine and cosine functions are used to decompose a periodic signal in a Fourier series expansion.

Note that for the Gaussian and log-normal distributions, Hermite polynomial is the best choice as they lead to exponential convergence rate [9]. For non Gaussian and non log-normal distributions, there are other orthogonal polynomials such as Legendre for uniform distribution, Charlier for Poisson distribution and Krawtchouk for Binomial distribution, etc. [8], [27].

For a random variable $v(t,\boldsymbol{\xi})$ with limited variance, where $\boldsymbol{\xi} = [\xi_1,\xi_2,\ldots,\xi_n]$ is a vector of zero mean orthonormal Gaussian random variables. The random variable can be approximated by truncated Hermite PC expansion as follows [9]:

$$v(t,\boldsymbol{\xi}) = \sum_{k=0}^{P} a_k H_k^n(\boldsymbol{\xi}) \tag{4}$$

where n is the number of independent random variables, $H_k^n(\boldsymbol{\xi})$ is n-dimensional Hermite polynomials and a_k are the deterministic coefficients. The number of terms P is given

$$P = \sum_{k=0}^{p} \frac{(n-1+k)!}{k!(n-1)!} \tag{5}$$

where p is the order of the Hermite PC. If only one random variable is considered, the 1-D Hermite polynomials are expressed as follows:

$$H_0^1(\xi) = 1, H_1^1(\xi) = \xi, H_2^1(\xi) = \xi^2 - 1, H_3^1(\xi) = \xi^3 - 3\xi, \ldots. \tag{6}$$

Hermite polynomials are orthogonal with respect to Gaussian weighted expectation (the superscript n is dropped for simple notation):

$$\langle H_i(\boldsymbol{\xi}), H_j(\boldsymbol{\xi})\rangle = \left\langle H_i^2(\boldsymbol{\xi})\right\rangle \delta_{ij} \tag{7}$$

where δ_{ij} is the Kronecker delta and $\langle *,*\rangle$ denotes an inner product defined as follows:

$$\langle f(\boldsymbol{\xi}), g(\boldsymbol{\xi})\rangle = \frac{1}{\sqrt{(2\pi)^n}} \int f(\boldsymbol{\xi}) g(\boldsymbol{\xi}) e^{-\frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{\xi}} d\boldsymbol{\xi}. \tag{8}$$

Like Fourier series, the coefficient a_k can be found by a projection operation onto the Hermite PC basis

$$a_k(t) = \frac{\langle v(t,\boldsymbol{\xi}), H_k(\boldsymbol{\xi})\rangle}{\langle H_k^2(\boldsymbol{\xi})\rangle} \qquad \forall k \in \{0,\ldots,P\}. \tag{9}$$

B. Simulation Approach Based on Hermite PCs

To simplify the presentation, we first assume that C and G are deterministic in (3). We will remove this assumption later. In case that $v(t,\boldsymbol{\xi})$ is unknown random variable vector (with unknown distributions) like node voltages in (3), then the coefficients can be computed by using Galerkin method, which states that the best approximation of $v(t,\boldsymbol{\xi})$ is obtained when the error $\Delta(t,\boldsymbol{\xi})$, which is defined as

$$\Delta(t,\boldsymbol{\xi}) = Gv(t) + C\frac{dv(t)}{dt} - I\left(t,\boldsymbol{\xi}(\theta)\right) \tag{10}$$

is orthogonal to the Hermite polynomials. That is

$$\langle \Delta(t,\boldsymbol{\xi}), H_k(\boldsymbol{\xi})\rangle = 0, \qquad i = 0,1,\ldots,P. \tag{11}$$

In this way, we transform the stochastic analysis process to a deterministic process, where we only need to compute the coefficients of its Hermite PC. Once we obtain those coefficients, the mean and variance of the random variables can be easily computed as shown later in the section.

For illustration purpose, considering one Gaussian variable $\boldsymbol{\xi} = [\xi_1]$ and we then can assume that the node voltage response can be written as a second-order ($p = 2$) Hermite PC

$$v(t,\boldsymbol{\xi}) = v_0(t) + v_1(t)\xi_1 + v_2(t)\left(\xi_1^2 - 1\right) \tag{12}$$

assuming that the input leakage current sources can also be represented by a second Hermite PC

$$I(t,\boldsymbol{\xi}) = I_0(t) + I_1(t)\xi_1 + I_2(t)\left(\xi_1^2 - 1\right). \tag{13}$$

By applying the Galerkin equation (11) and note the orthogonal property of the various orders of Hermite PCs, we end up with the following equations:

$$Gv_i(t) + C\frac{dv_i(t)}{dt} = I_i(t) \tag{14}$$

where $i = 0,1,2,\ldots,P$.

For two independent Gaussian variables, we have

$$\begin{aligned} v(t,\boldsymbol{\xi}) = v_0(t) + v_1(t)\xi_1 + v_2(t)\xi_2 + v_3(t)\left(\xi_1^2 - 1\right) \\ + v_4(t)\left(\xi_2^2 - 1\right) + v_5(\xi_1\xi_2). \end{aligned} \tag{15}$$

Assuming that we have a similar second-order Hermite PC for input leakage current $I(t,\boldsymbol{\xi})$,

$$\begin{aligned} I(t,\boldsymbol{\xi}) = I_0(t) + I_1(t)\xi_1 + I_2(t)\xi_2 + I_3(t)\left(\xi_1^2 - 1\right) \\ + I_4(t)\left(\xi_2^2 - 1\right) + I_5(\xi_1\xi_2). \end{aligned} \tag{16}$$

The (14) is valid with $i = 0,\ldots,5$. For more (more than two) Gaussian variables, we can obtained the similar results with more coefficients of Hermite PCs to be solved by using (14).

Once we obtain the Hermite PC of $v(t,\boldsymbol{\xi})$, we can obtain the mean and variance of $v(t,\boldsymbol{\xi})$ trivially as (one Gaussian variable case)

$$\begin{aligned} E\left(v(t,\boldsymbol{\xi})\right) &= v_0(t) \\ Var\left(v(t,\boldsymbol{\xi})\right) &= v_1^2(t)Var(\xi_1) + v_2^2(t)Var\left(\xi_1^2 - 1\right) \\ &= v_1^2(t) + 2v_2^2(t). \end{aligned} \tag{17}$$

One critical problem remains so far is how to obtain the Hermite PC (13) for leakage current with log-normal distribution. This will be explained in details in the next section.

IV. Hermite PCs for Log-Normal Leakage Current Variations

In this section, we present the new method for representing the log-normal leakage current distributions by using Hermite PCs with one or more independent Gaussian variables representing the channel length or threshold voltage variations.

Our method is based on [8] and we will show how it can be applied to solve our problems for one or more independent Gaussian variables.

A. Hermite PC Representation of Log-Normal Variables

Let $g(\boldsymbol{\xi})$ be the Gaussian random variable, denoting threshold voltage or device channel length. Let $l(\boldsymbol{\xi})$ be the random variable obtained by taking the exponential of $g(\boldsymbol{\xi})$

$$l(\boldsymbol{\xi}) = e^{g(\boldsymbol{\xi})}, g(\boldsymbol{\xi}) = ln\left(l(\boldsymbol{\xi})\right). \tag{18}$$

Obviously, for the MOS device leakage current equation (1), leakage current, $I_{\text{off}} = cI_l(V_{\text{th}}) = ce^{-V_{\text{th}}}$, where the leakage component $I_l(V_{\text{th}})$ is a log-normal random variable. Let the mean and the variance of $g(\boldsymbol{\xi})$ as μ_g and σ_g^2, then the mean and variance of $l(\boldsymbol{\xi})$ are

$$\mu_l = e^{\left(\mu_g + \frac{\sigma_g^2}{2}\right)} \tag{19}$$

$$\sigma_l^2 = e^{\left(2\mu_g + \sigma_g^2\right)}\left[e^{\sigma_g^2} - 1\right] \tag{20}$$

respectively.

For a general Gaussian variable $g(\boldsymbol{\xi})$, it can always be represented in the following affine form:

$$g(\boldsymbol{\xi}) = \sum_{i=0}^{n} \xi_i g_i \tag{21}$$

where ξ_i are orthonormal Gaussian variables. i.e., $\langle \xi_i\xi_j\rangle = \delta_{ij}$, $\langle \xi_i\rangle = 0$ and $\xi_0 = 1$ and g_i is the coefficient of the individual Gaussian variables. Note that such form can always be obtained by using Karhunen–Loeve orthogonal expansion method [9].

In our problem, we need to represent the log-normal random variable $l(\boldsymbol{\xi})$ by using the Hermite PC expansion form

$$l(\boldsymbol{\xi}) = \sum_{k=0}^{P} l_k H_k^n(\boldsymbol{\xi}) \tag{22}$$

where $l_0 = \exp[\mu_g + (\sigma_g^2/2)]$. To find the other coefficients, we can apply (9) on $l(\boldsymbol{\xi})$. Therefore, we have

$$l_k(t) = \frac{\langle l(t,\boldsymbol{\xi}), H_k(\boldsymbol{\xi})\rangle}{\langle H_k^2(\boldsymbol{\xi})\rangle} \qquad \forall k \in \{0,\ldots,P\}. \tag{23}$$

It was shown in [8], $l(\boldsymbol{\xi})$ can be written as

$$l(\boldsymbol{\xi}) = \frac{\langle H_k(\boldsymbol{\xi}-\mathbf{g})\rangle}{\langle H_i^2(\boldsymbol{\xi})\rangle} = \exp\left[\mu_g + \frac{1}{2}\sum_{j=1}^{n} g_j^2\right] \tag{24}$$

where n is the number of independent Gaussian random variables.

The log-normal process can then be written as

$$l(\boldsymbol{\xi}) = l_0\left(1 + \sum_{i=1}^{n}\boldsymbol{\xi}_i g_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(\boldsymbol{\xi}_i\boldsymbol{\xi}_j - \delta_{ij})}{\langle(\boldsymbol{\xi}_i\boldsymbol{\xi}_j - \delta_{ij})^2\rangle} g_i g_j + \ldots\right) \tag{25}$$

where g_i is defined in (21).

B. Hermite PC Representation With One Gaussian Variable

In this case, $\boldsymbol{\xi} = [\xi_1]$. For the second-order Hermite PC ($P = 2$), following (25), we have

$$l(\boldsymbol{\xi}) = l_0\left(1 + \sigma_g\xi_1 + \frac{1}{2}\sigma_g^2\left(\xi_1^2 - 1\right)\right). \tag{26}$$

Hence, the desired Hermite PC coefficients $I_{0,1,2}$ can be expressed as l_0, $l_0\sigma_g$ and $(1/2)l_0\sigma_g^2$, respectively.

C. Hermite PC Representation of Two and More Gaussian Variables

For two random variables ($n = 2$), assume that $\boldsymbol{\xi} = [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2]$ is a normalized uncorrelated Gaussian random variable vector that represents random variable $g(\boldsymbol{\xi})$

$$g(\boldsymbol{\xi}) = \mu_g + \sigma_1\boldsymbol{\xi}_1 + \sigma_2\boldsymbol{\xi}_2. \tag{27}$$

Note that

$$\left\langle(\xi_i\xi_j - \delta_{ij})^2\right\rangle = \left\langle\xi_i^2\xi_j^2\right\rangle = \left\langle\xi_i^2\right\rangle\left\langle\xi_j^2\right\rangle = 1.$$

Therefore, the expansion of the log-normal random variables using second-order Hermite PCs can be expressed as

$$l(\boldsymbol{\xi}) = l_0\left(1 + \sigma_1\xi_1 + \sigma_2\xi_2 + \frac{\sigma_1^2}{2}\left(\xi_1^2 - 1\right) + \frac{\sigma_2^2}{2}\left(\xi_2^2 - 1\right) + 2\sigma_1\sigma_2\xi_1\xi_2\right) \tag{28}$$

where

$$\mu_l = l_0 = \exp\left(\mu_g + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2\right).$$

Hence, the desired Hermite PC coefficients, $I_{0,1,2,3,4,5}$, can be expressed as l_0, $l_0\sigma_1$, $l_0\sigma_2$, $(1/2)l_0\sigma_1^2$, $(1/2)l_0\sigma_2^2$, and $2l_0\sigma_1\sigma_2$ respectively.

Similarly, for four Gaussian random variables, assume that $\boldsymbol{\xi} = [\xi_1, \xi_2, \xi_3, \xi_4]$ is a normalized, uncorrelated Gaussian random variable vector. The random variable $g(\boldsymbol{\xi})$ can be expressed as

$$g = \mu_g + \sum_{i=1}^{4}\sigma_i\boldsymbol{\xi}_i. \tag{29}$$

As a result, the log-normal random variable $l(\boldsymbol{\xi})$ can be expressed as

$$l(\boldsymbol{\xi}) = l_0\left(1 + \sum_{i=1}^{4}\boldsymbol{\xi}_i\sigma_i + \sum_{i=1}^{4}\frac{1}{2}\left(\boldsymbol{\xi}_i^2 - 1\right)\sigma_i^2 + \sum_{i=1}^{4}\sum_{j=1}^{4}\boldsymbol{\xi}_i\boldsymbol{\xi}_j\sigma_i\sigma_j + \ldots\right) \tag{30}$$

where

$$\mu_l = l_0 = \exp\left(\sigma_0 + \frac{1}{2}\sum_{i=1}^{4}\sigma_i^2\right).$$

Hence, the desired Hermite PC coefficients can be expressed using the (30) above.

Once we have the Hermite PC representation of the leakage current sources $I(t,\boldsymbol{\xi})$, the node voltages $v(t,\boldsymbol{\xi})$ can be computed by using (14) with proper order p of the PCs to obtain all the Hermite PC coefficients of $v(t,\boldsymbol{\xi})$.

V. Spatial Correlation

In this section, we consider the spatial correlation among different variations within a die. Spatial correlations exist in the intra-die variations in different forms and have been modeled for timing analysis [3], [18]. The general way to consider spatial correlation is by means of mapping the correlated random variables into a set of independent variables. This can be done by using some orthogonal mapping techniques, such as principal component analysis(PCA). In this paper, we also apply PCA method in our spectral statistical analysis framework for power/grid statistical analysis.

A. Concept of Principal Component Analysis

We first briefly review the concept of principal component analysis, which is used here to transform the random variables with correlation to uncorrelated random variables [12].

Suppose that x is a vector of n random variables, $x = [x_1, x_2, \ldots, x_n]^T$, with covariance matrix C and mean vector $\mu_x = [\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_n}]$. To find the orthogonal random variables, we first calculate the eigenvalue and corresponding eigenvector. Then, by ordering the eigenvectors in

descending-order eigenvalues, the orthogonal matrix A will be obtained. Here, A is expressed as

$$A = \left[e_1^T, e_2^T, \ldots, e_n^T\right]^T \tag{31}$$

where e_i is the corresponding eigenvector to eigenvalue λ_i, which satisfies

$$\lambda_i e_i = C e_i, \qquad i = 1, 2, \ldots, n \tag{32}$$

and

$$\lambda_i < \lambda_{i-1}, \qquad i = 2, 3, \ldots, n. \tag{33}$$

With A, we can perform the transformation to get orthogonal random variables y, $y = [y_1, y_2, \ldots, y_n]^T$ by using

$$y = A(x - \mu_x) \tag{34}$$

where, y_i is a random variable with Gaussian distribution. The mean, μ_{y_i}, is 0 and the standard deviation, σ_{y_i}, is $\sqrt{\lambda_i}$ on the condition that [12]

$$e_i^T e_i = 1, \qquad i = 1, 2, \ldots, n. \tag{35}$$

Here, because of the orthogonal property of matrix A

$$A^{-1} = A^T. \tag{36}$$

To reconstruct the original random variables, we use the following equation:

$$x = A^T y + \mu_x. \tag{37}$$

B. Spatial Correlation in Statistical Power Grid Analysis

To consider intra-die variation in $V_{\rm th}$, the chip is divided into n regions. Assuming $\Phi = [\Phi_1, \Phi_2, \ldots, \Phi_n]$ is a random variable vector, representing the variation of $V_{\rm th}$ on different part of the circuit. In other words, in the ith region, the leakage current $I_{off_i} = ce^{V_{\rm th}(\Phi_i)}$, follows the log-normal distribution. Here, Φ_i is a random variable with Gaussian distribution. $\mu_\Phi = [\mu_{\Phi_1}, \mu_{\Phi_2}, \ldots, \mu_{\Phi_n}]$ is the mean vector of Φ and C is the covariance matrix of Φ.

With PCA, we can get the corresponding uncorrelated random variables $\phi = [\phi_1, \phi_2, \ldots, \phi_n]$ from the equation

$$\phi = A(\Phi - \mu_\Phi). \tag{38}$$

Also, the original random variables can be expressed as

$$\Phi_i = \sum_{j=1}^{n} a_{ij}\phi_j + \mu_{\Phi_i}, \qquad i = 1, 2, \ldots n \tag{39}$$

where a_{ij} is the ith row, jth column element in the orthogonal mapping matrix defined in (34). $\phi = [\phi_1, \phi_2, \ldots, \phi_n]$ is a vector with orthogonal Gaussian random variables. The mean of ϕ_j is 0 and variance is λ_j, $j = 1, 2, \ldots, n$. The distribution of ϕ_i can be written as

$$\phi_i = \mu_{\phi_i} + \sigma_{\phi_i}\hat{\xi}_i, \qquad i = 1, 2, \ldots, n \tag{40}$$

where $\hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n]$ is a vector with orthogonal normal Gaussian random variable. Φ_i can be expressed with normal random variables, $\hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n]$:

$$\Phi_i = \sum_{j=1}^{n} a_{ij}\sqrt{\lambda_j}\hat{\xi}_j + \mu_{\Phi_i}, \qquad i = 1, 2, \ldots, n. \tag{41}$$

With (41), the leakage current can be expanded as Hermite PC

$$\begin{aligned} I(\Phi_i) &\sim e^{\Phi_i} \\ &= e^{\sum_{j=1}^{n} g_j\hat{\xi}_j + \mu_{\Phi_i}} \\ &= \mu_i \left(1 + \sum_{j=1}^{n} \hat{\xi}_j g_j \right. \\ &\qquad \left. + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{(\hat{\xi}_j\hat{\xi}_k - \delta_{jk})}{\left\langle (\hat{\xi}_j\hat{\xi}_k - \delta_{jk})^2 \right\rangle} g_j g_k + \ldots \right). \end{aligned} \tag{42}$$

Here

$$g_j = a_{ij}\sqrt{\lambda_j}, \qquad j = 1, 2, \ldots, n. \tag{43}$$

Therefore, the MNA equation with correlated random variables Φ in current source can be expressed in terms of uncorrelated random variables $\hat{\xi}$ as follows:

$$Gv(t) + C\frac{dv(t)}{dt} = I_i(t, \hat{\xi}). \tag{44}$$

With orthogonal property of $\hat{\xi}$, (44) will be simply solved by using (14), $i = 1, 2, \ldots, P$.

VI. Variations in Wires and Leakage Currents

In this section, we will consider variations in width (W), thickness (T) of wires of power grids, as well as threshold voltage $(V_{\rm th})$ in active devices which are reflected in the leakage currents. Meanwhile, without loss of generality, these variations are supposed to be independent of each other. As

mentioned in [10], the MNA equation for the ground circuit will become

$$G(\xi_g)v(t) + C(\xi_c)\frac{dv(t)}{dt} = I(\xi_I, t). \tag{45}$$

The variation in width W and thickness T will cause variation in conductance matrix G and capacitance matrix C while variation in threshold voltage will cause variation in leakage currents I. Thus, the conductance and capacitance of wires can be expressed as in [10]

$$\begin{aligned} G(\xi_g) &= G_0 + G_1\xi_g \\ C(\xi_c) &= C_0 + C_1\xi_c \end{aligned} \tag{46}$$

where G_0, C_0 represents the deterministic components of conductance and capacitance of the wires. G_1, C_1 represents sensitivity matrices of the conductance and capacitance. ξ_g, ξ_c are normalized random variables with Gaussian distribution, representing process variation in wires of conductance and capacitor, respectively. As mentioned in previous section, the variation in leakage current can be represented by a second Hermite PC as in (26)

$$I(t, \xi_I) = I_0(t) + I_1(t)\xi_I + I_2(t)\left(\xi_I^2 - 1\right) \tag{47}$$

where ξ_I is a normalized Gaussian distribution random variable representing variation in threshold voltage. $I(t, \xi_I)$ follows log-normal distribution as

$$\begin{aligned} I &= e^{g(\xi_I)} \\ g(\xi_I) &= \mu_I + \sigma_I\xi_I. \end{aligned} \tag{48}$$

As in previous part, the desired Hermite PC coefficients, $I_{0,1,2}$, can be expressed as I_0, $I_0\sigma_I$ and $(1/2)I_0\sigma_I^2$ respectively. I_0 is the mean of leakage current source, which is expressed as

$$I_0 = exp\left(\mu_I + \frac{1}{2}\sigma_I^2\right). \tag{49}$$

Considering the influence of ξ_g, ξ_c, ξ_I, the node voltage is therefore expended by Hermite PC in the second-order form as

$$\begin{aligned} v(t,\xi) = {} & v_0(t) + v_1(t)\xi_g + v_2(t)\xi_c + v_3(t)\xi_I \\ & + v_4(t)\left(\xi_g^2 - 1\right) + v_5(t)\left(\xi_c^2 - 1\right) + v_6(t)\left(\xi_I^2 - 1\right) \\ & + v_7(t)\xi_g\xi_c + v_8(t)\xi_g\xi_I + v_9(t)\xi_c\xi_I. \end{aligned} \tag{50}$$

Now, the task is to compute coefficients of the Hermite PC of node voltage $v(t, \xi)$. Applying Galerkin equation (11), we only need to solve the equations as follows:

$$\begin{aligned} \langle\Delta(t,\boldsymbol{\xi}), 1\rangle &= 0; \quad \langle\Delta(t,\boldsymbol{\xi}), \boldsymbol{\xi}_g\rangle = 0 \\ \langle\Delta(t,\boldsymbol{\xi}), \boldsymbol{\xi}_c\rangle &= 0; \quad \langle\Delta(t,\boldsymbol{\xi}), \boldsymbol{\xi}_I\rangle = 0 \\ \left\langle\Delta(t,\boldsymbol{\xi}), \boldsymbol{\xi}_g^2 - 1\right\rangle &= 0; \quad \left\langle\Delta(t,\boldsymbol{\xi}), \boldsymbol{\xi}_c^2 - 1\right\rangle = 0 \\ \left\langle\Delta(t,\boldsymbol{\xi}), \boldsymbol{\xi}_I^2 - 1\right\rangle &= 0; \quad \langle\Delta(t,\boldsymbol{\xi}), \boldsymbol{\xi}_g\boldsymbol{\xi}_c\rangle = 0 \\ \langle\Delta(t,\boldsymbol{\xi}), \boldsymbol{\xi}_g\boldsymbol{\xi}_I\rangle &= 0; \quad \langle\Delta(t,\boldsymbol{\xi}), \boldsymbol{\xi}_c\boldsymbol{\xi}_I\rangle = 0. \end{aligned} \tag{51}$$

With the distribution of $\boldsymbol{\xi}_g, \boldsymbol{\xi}_c, \boldsymbol{\xi}_I$, we can get these coefficients $v(t) = [v_0(t), v_1(t), \ldots, v_9(t)]^T$ of node voltage as

$$\widetilde{G}v(t) + \widetilde{C}\frac{dv(t)}{dt} = \widetilde{I}(t) \tag{52}$$

where

$$\widetilde{G} = \begin{bmatrix} G_0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ G_1 & G_0 & 0 & 0 & 2G_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_0 & 0 & 0 & 0 & 0 & G_1 & 0 & 0 \\ 0 & 0 & 0 & G_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_1 & 0 & 0 & G_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_0 & 0 & 0 \\ 0 & 0 & 0 & G_1 & 0 & 0 & 0 & 0 & G_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_0 \end{bmatrix}$$

$$\widetilde{C} = \begin{bmatrix} C_0 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_0 & 0 & 0 & 0 & 0 & 0 & C_1 & 0 & 0 \\ C_1 & 0 & C_0 & 0 & 0 & 2C_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_1 & 0 & 0 & C_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_0 & 0 \\ 0 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 & C_0 \end{bmatrix}$$

$$\widetilde{I}(t) = [I_0(t), 0, 0, I_1(t), 0, 0, I_2(t), 0, 0, 0]^T. \tag{53}$$

Knowing Hermite PC coefficients of node voltage $v(t, \xi)$, it is easy to get the mean and variance of $v(t, \xi)$, which describe the random characteristic of node voltage in the given circuit.

We remark that the proposed method will lead to large circuit matrices, which will add more computation costs. To mitigate this scalability problem, for really large power grid circuits, we can apply partitioning strategies to compute the variational responses for each subcircuits, which will be small enough for efficient computation, as done in the existing work [4], [32].

VII. Experimental Results

This section describes the simulation results of circuits with log-normal leakage current distributions for a number of power grid networks. All the proposed methods have been implemented in Matlab. Sparse techniques are used in the Matlab. All the experimental results are carried out in a Linux system with dual Intel Xeon CPUs with 3.06 Ghz and 1 GB memory.

The power grid circuits we test are *RC* mesh circuits based on the values from some industry circuits, which are driven by only leakage currents as we are only interested in the variations from the leakage currents. The resistor values are in the range 10^{-2} Ω and capacitor values are in the range of 10^{-12} farad.

A. Comparison With Taylor Expansion Method

We first compare the proposed method with the simple Taylor expansion method for one and more Gaussian variables.

TABLE I
ACCURACY COMPARISON BETWEEN HERMITE PC (HPC) AND TAYLOR EXPANSION

δ_g	0.01	0.1	0.3	0.5	0.7
HPC (%)	3.19	1.88	2.07	5.5	2.92
Taylor (%)	3.19	1.37	2.41	16.6	24.02

For simplicity, we assume one Gaussian random variable $g(\xi)$, which is expressed as

$$g = \mu_g + \sigma_g \boldsymbol{\xi} \tag{54}$$

where $\boldsymbol{\xi}$ is a normalized Gaussian random variable with $\langle \boldsymbol{\xi} \rangle = 0$, and $\langle \boldsymbol{\xi}^2 \rangle = 1$.

The log-normal random variable $l(\boldsymbol{\xi})$, obtained from $g(\boldsymbol{\xi})$, is written as

$$l(\boldsymbol{\xi}) = e^{g(\boldsymbol{\xi})} = \exp(\mu_g + \sigma_g \boldsymbol{\xi}). \tag{55}$$

Expand the exponential into Taylor series and keep all the terms up to second order, then we have

$$\begin{aligned} l(\boldsymbol{\xi}) &= 1 + \sum_{i=0}^{1} \boldsymbol{\xi}_i g_i + \frac{1}{2} \sum_{i=0}^{1} \sum_{j=0}^{1} \boldsymbol{\xi}_i \boldsymbol{\xi}_j g_i g_j + \dots \\ &= 1 + \mu_g + \frac{1}{2}\mu_g^2 + \frac{1}{2}\sigma_g^2 + (\sigma_g + \mu_g \sigma_g)\boldsymbol{\xi} \\ &\quad + \frac{1}{2}\sigma_g^2(\boldsymbol{\xi}^2 - 1) + \cdots\cdots. \end{aligned} \tag{56}$$

We observe that the second-order Taylor expansion, as shown in (56), is similar to second-order Hermite PC in (28). Hence, the Galerkin method can still be applied, we then use (14) to obtain the Hermite PC coefficients of node voltage $v(t, \boldsymbol{\xi})$ accordingly. We want to emphasize, however, that the polynomials generated by Taylor expansion in general are not orthogonal with respect to Gaussian distributions and can't be used with Galerkin method, unless we only keep the first order of Taylor expansion results (with less accuracy). In this case, the resulting node voltage distribution is still Gaussian, which obviously is not correct.

We note that the first-order Taylor expansion has been used in the statistic timing analysis [3]. The delay variations, owning to interconnects and devices, can be approximated with this limitation. The skew distributions may be computed easily with Gaussian process.

To compare these two methods, we use the MC method to measure the accuracies of two methods in terms of standard deviation. For MC, we sample 2000 times, which represents 97.7% accuracy. The results are summarized in Table I. In this table, δ_g is the standard deviation of the Gaussian random threshold voltage Gaussian variable in the log-normal current source, *HPC* is the standard deviation from the Hermite PC method in terms of relative percentage against the MC method. *Taylor* is the standard deviation from the Taylor expansion method in terms of relative percentage against the MC method.

We can observe that when the variation of current source increases, the Taylor expansion method will result in significant errors compared to the MC method. While the proposed method has the smaller errors for all cases. This clearly shows the advantage of the proposed method.

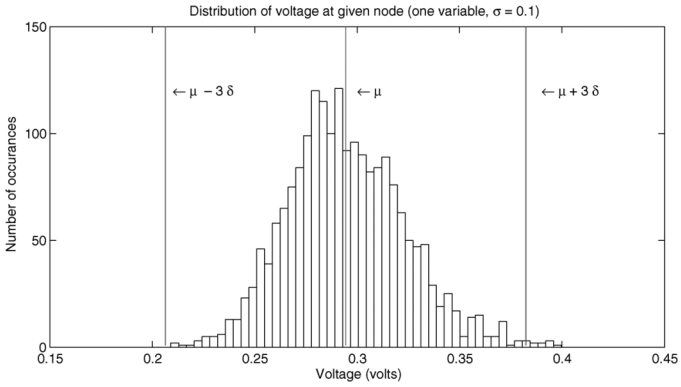

Fig. 1. Distribution of the voltage in a given node with one Gaussian variable, $\sigma_g = 0.1$, at time 50 ns when the total simulation time is 200 ns.

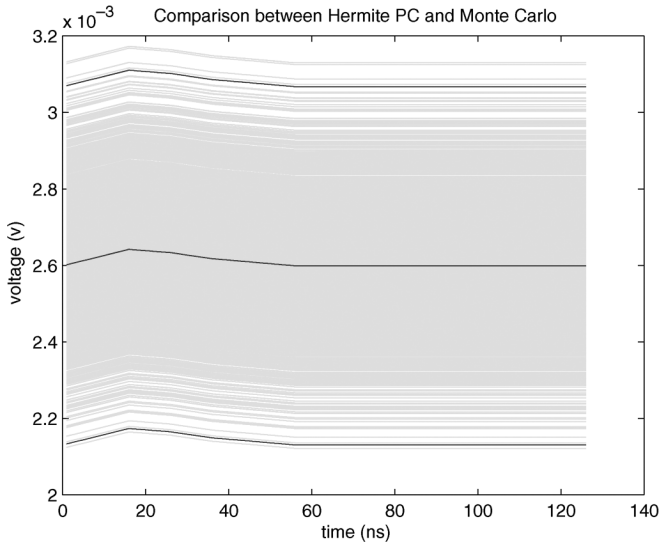

Fig. 2. Distribution of the voltage caused by the leakage currents in a given node with one Gaussian variable, $\sigma_g = 0.5$, in the time instant from 0 to 126 ns.

B. Examples Without Spatial Correlation

Fig. 1 shows the node voltage distributions at one node on a certain point of a ground network with 1720 nodes. The MC results are obtained by 2000 samples. The standard deviations of the log-normal current sources with one Gaussian variable is 0.1. The mean and 3σ computed by the Hermite PC method are also marked in the figure which fits very well with the MC results. Fig. 2 shows the node voltages and its variations caused by the leakage currents from 0 ns to 126 ns. The circuit selected contains 64 nodes with one Gaussian variable of 0.06 in the current source. The blue dotted lines are mean, upper bound and lower bound. The cyan lines are node voltages of MC with 2000 times. Most of the MC results are in between upper bound and lower bound.

Another observation is that when standard deviation, σ_g, is small, the shape looks like Gaussian as in Fig. 1, but it is log-normal indeed. In the case of two random variables with one

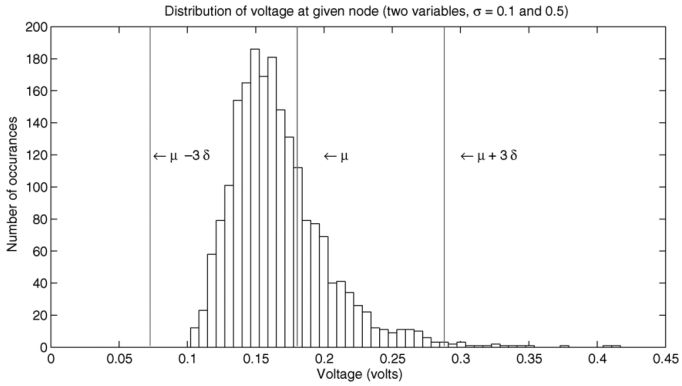

Fig. 3. Distribution of the voltage in a given node with two Gaussian variables, $\sigma_{g1} = 0.1$ and $\sigma_{g2} = 0.5$, at time 50 ns when the total simulation time is 200 ns.

TABLE II
CPU TIME COMPARISON WITH THE MC METHOD OF ONE RANDOM VARIABLE

Ckt	#node	p	n	MC (s)	#MC	HPC (s)	Speedup
gridrc_6	280	2	1	766.06	2000	1.0156	754.3
gridrc_12	3240	2	1	4389	2000	8.3281	527.0
gridrc_5	49600	2	1	2.3×10^5	2000	298.02	771.76

TABLE III
CPU TIME COMPARISON WITH THE MC METHOD OF TWO RANDOM VARIABLES

Ckt	#node	p	n	MC (s)	#MC	HPC (s)	Speedup
gridrc_3	280	2	2	1.05×10^3	2000	2.063	507.6
gridrc_5	49600	2	2	2.49×10^5	2000	445.6	558.7
gridrc_9	105996	2	2	6.11×10^5	2000	1141.8	535.1

large and the other small standard deviations, the larger one dominates, which shows the shape of log-normal as in Fig. 3.

To consider multiple random variables, we divide the circuit into several partitions. We first divide the circuit into two parts. Fig. 3 shows the node voltage of one node of a particular time instance of a ground network with 336 nodes with two independent variables. The standard deviations for two Gaussian variations are $\sigma_{g1} = 0.5$, $\sigma_{g2} = 0.1$. The 3σ variations are also marked in the figure.

Table II and Table III shows the speedup of the Hermite PC method over MC method with 2000 samples considering one and two random variables, respectively.

In two tables, *#node* is the number of nodes in the power grid circuits. p is the order of the Hermite PCs and n is the number of independent Gaussian random variables. *#MC* is the number of samples used for MC method. *HPC* and *MC* represent the CPU times used for Hermite PC method and MC method respectively. It can be seen that the proposed method is about two order of magnitude faster than the MC method.

When more Gaussian variables are used for modeling intra-die variations, we need more Hermite PC coefficients to compute. Hence, the speedup will be smaller if the MC method uses the same number of samples as shown in *gridrc_12*. Also, one observation is that the speedup depends on the sampling size in MC method. The speedup of the proposed method over the MC method depends on many factors such as the order of polynomials, number of variables, etc. In general, speedup

$\Phi_1 = \xi_1 + 0.5\xi_2$	$\Phi_2 = \xi_2 + 0.5\xi_1$

Fig. 4. Correlated random variables setup in ground circuit divided into two parts.

TABLE IV
COMPARISON BETWEEN NON-PCA AND PCA AGAINST MC METHODS

ckt	#nodes	Mean		Std Dev	
		Non-PCA % error	PCA % error	Non-PCA % error	PCA % error
1	336	10.3	0.52	18.8	1.13
2	645	8.27	0.59	11.4	1.16
3	1160	10.8	0.50	2.6	0.73

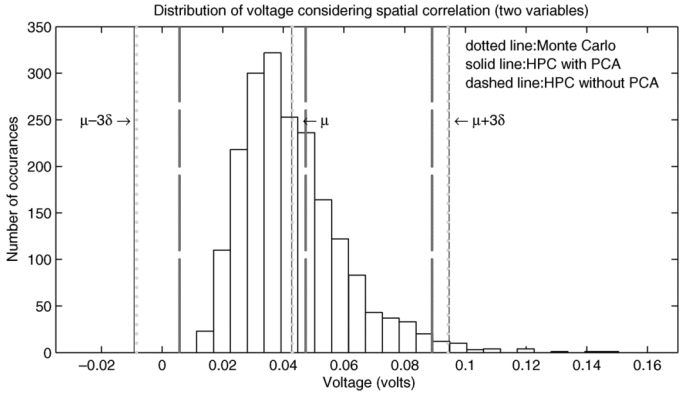

Fig. 5. Distribution of the voltage in a given node with two Gaussian variables with spatial correlation, at time 70 ns when the total simulation time is 200 ns.

should not have a clear relationship with the circuit sizes. We still use 2000 samples for MC, which represent about 97.7% accuracy (as the error in MS is roughly $1/\sqrt{2000}$ for 2000 samples).

C. Examples With Spatial Correlation

To model the intra-die variations with spatial correlations, we divide the power grid circuit into several parts. We first consider that circuit is partitioned into two parts. In this case, we have two independent random current variables, ξ_1 and ξ_2. The correlated variables for the two parts are $\Phi_1 = \xi_1 + 0.5\xi_2$ and $\Phi_2 = \xi_2 + 0.5\xi_1$ respectively as shown in Fig. 4.

Table IV shows the error percentage of mean and standard deviation of the comparison between *MC* and *HPC* with Principal Component Analysis (PCA) and the comparison between *MC* and *HPC* without PCA. As shown in the table, it is necessary to use PCA when spatial correlation is considered. Fig. 5 shows the node voltage distribution of one certain node in a ground network with 336 nodes, using both PCA and non-PCA method.

To get more accuracy, we divide the circuit into four parts and each part has correlation with its neighbor as shown in Fig. 6. ϕ is the correlated random variable vector we use in the circuit. $\zeta = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]$ are independent Gaussian distribution random variables with standard deviations $\zeta_1 = 0.1$, $\zeta_2 = 0.2$, $\zeta_3 = 0.1$ and $\zeta_4 = 0.5$. Fig. 7 is the voltage distribution of a given node. The mean voltage and voltages of worst case are given as the dashed line. Fig. 8 is the voltage distribution of a circuit with 1160 nodes. The circuit is partitioned into 25 parts of

$\phi1=\zeta1+0.5\zeta2+0.5\zeta3$	$\phi3=\zeta3+0.5\zeta1+0.5\zeta4$
$\phi2=\zeta2+0.5\zeta1+0.5\zeta4$	$\phi4=\zeta4+0.5\zeta2+0.5\zeta3$

Fig. 6. Correlated random variables setup in ground circuit divided into four parts.

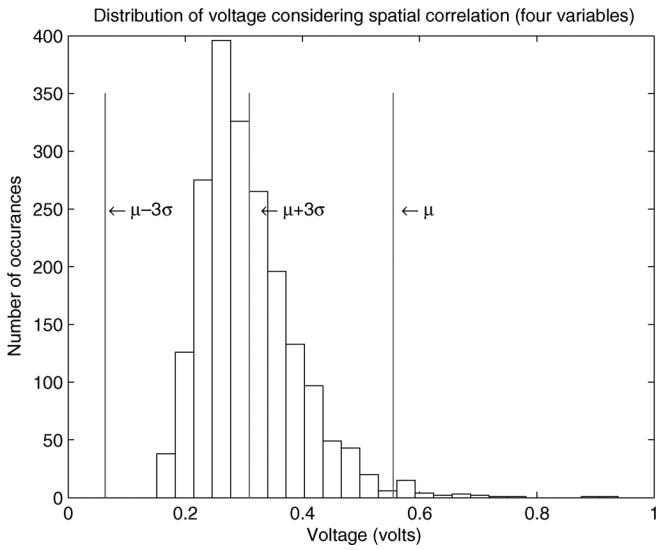

Fig. 7. Distribution of the voltage in a given node with four Gaussian variables with spatial correlation, at time 30 ns when the total simulation time is 200 ns.

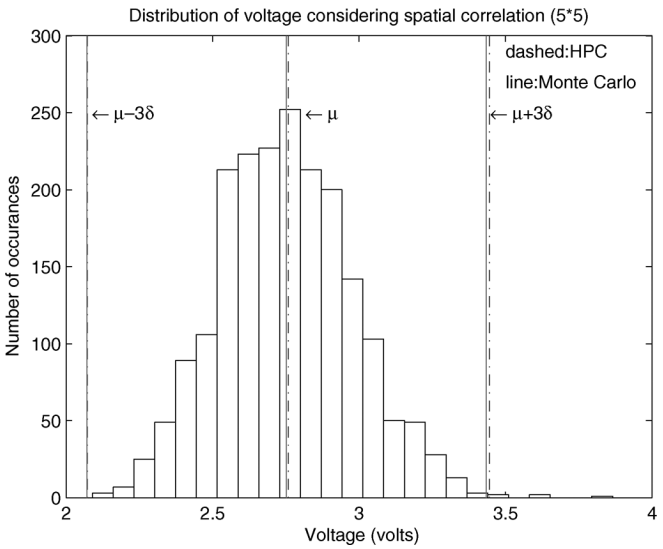

Fig. 8. Distribution of the voltage in a given node with circuit partitioned of $5 * 5$ with spatial correlation, at time 30 ns when the total simulation time is 200 ns.

five rows and five columns with spatial correlation. The dashed blue lines are mean, upper bound and lower bound by Hermite PC. While the solid red lines are mean, upper bound and lower bound by MC of 2000 times.

Note that the size of the ground networks we analyzed is mainly limited by the solving capacity of Matlab on a single Intel CPU Linux workstation. Given long simulation time of large MC sampling runs, we limit the ground network size to about 3000 nodes.

Also note that for more accurate modeling, we need to have more partitions of the circuits and thus more independent Gaussian variables are needed as shown in [3].

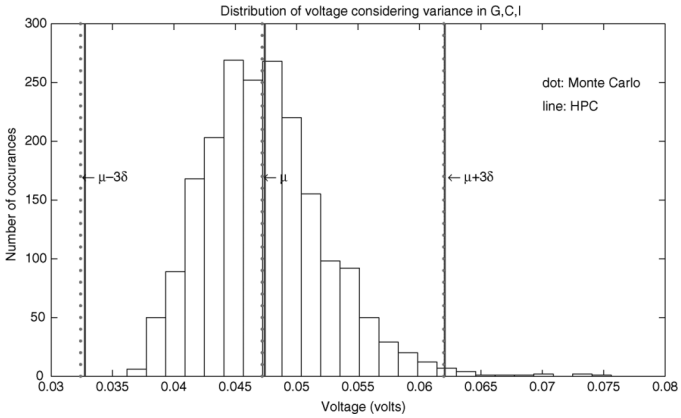

Fig. 9. Distribution of the voltage in a given node in circuit5 with variation on G,C,I, at time 50 ns when the total simulation time is 200 ns.

TABLE V
CPU TIME COMPARISON WITH THE MC METHOD CONSIDERING VARIATION IN G, C, I

Ckt	#node	MC (s)	HPC (s)	Speedup
gridrc_6	280	1320.1	9.25	142.7
gridrc_12	3240	12183	141.4	86.2
gridrc_62	9964	63832	3261	19.6

D. Consideration of Variations in Both Wire and Currents

Considering variation in conductance, capacitor and leakage current, Fig. 9 shows the node voltage distribution at one node of ground circuit, Circuit4, which contains 280 nodes. The maximum 3δ variation are 10% in ξ_g, ξ_c and ξ_I. In the figures, the solid lines are the mean voltage and worst case voltages using HPC method. The histogram bars are the MC results of 2000 samples. The dotted lines are the mean voltage and worst case voltage of the 2000 samples. From the figures we can see that results getting from two methods match very well.

Table V shows the CPU speedup of *HPC* method over *MC* method. The sample numbers of MC is 3500 and we can see that the proposed method is about two orders of magnitudes faster than the MC method when considering variations in conductance, capacitors and voltage sources. The speedup becomes smaller for larger circuits. This is because the super-linear time complexity of linear solver as the augmented matrices in (53) grow faster than each individual matrices G_i and C_i. The proposed method does not favor very large circuits. Practically, this scalability problem can be mitigated by using partitioning-based strategies [4].

VIII. CONCLUSION

In this paper, we have proposed a new stochastic simulation method for fast estimating the voltage variations from the process-induced log-normal leakage current variations with spatial correlations. The new analysis is based on the Hermite PC representation of random processes. We extended the existing

Hermite PC based power grid analysis method [10] by considering log-normal leakage distributions as well as the consideration of the spatial correlations. The new method considers both log-normal leakage distribution and wire variations at the same time. Our experimental results show that the new method is more accurate than the Gaussian-only Hermite PC using the Taylor expansion method for analyzing leakage current variations and two orders of magnitude faster than MC methods with small variation errors. In the presence of spatial correlations, method without considering the spatial correlations may lead to large errors, roughly 8%-10% in our tested cases, if correlation is not considered. Experimental results show the correctness and high accuracy of the proposed method. It leads to about 1% or less of errors in both mean and standard deviations and is about two orders of magnitude faster than MC methods.

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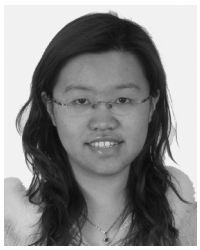

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