SBPOR: Second-Order Balanced Truncation for Passive Order Reduction of RLC Circuits *

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ABSTRACT

RLC circuits have been shown to be better formulated as second-order systems instead of first-order systems. The corresponding model order reduction techniques for second-order systems have been developed. However, existing techniques are mainly based on moment-matching concept. While suitable for the reduction of large-scale circuits, those approaches cannot generate reduced models as compact as desired. To achieve smaller models with better error control, a novel technique, SBPOR (Second-order Balanced truncation for Passive Order Reduction), is proposed in this paper, which is the first second-order balanced truncation method proposed for passive reduction of RLC circuits. SBPOR is superior to the pioneering work in the control community because second-order systems can be balanced via congruency transformation without any accuracy loss. In addition, compared with the first-order balanced truncation approaches, SBPOR is a better choice for RLC reduction. SBPOR preserves not only passivity but also the structure information inherent to RLC circuits, which is a special need for RLC reduction. In addition, SBPOR is computationally more efficient as it only needs to solve one linear matrix equation instead of two quadratic matrix equations.

Categories and Subject Descriptors

I.6.5 [**Simulation and Modeling**]: Model Development—*modeling methodologies*

General Terms

Algorithms

Keywords

Model order reduction, second-order balanced truncation

1. INTRODUCTION

Model order reduction (MOR) is an efficient technique to reduce the circuit complexity while producing a good approximation of the input and output behavior. From formulation point of view, MOR techniques can be classified into first-order based methods (using modified nodal analysis, MNA) and second-order based methods (using nodal analysis, NA). In terms of projection subspace, these approaches are divided into two broad categories, namely moment matching based methods and balanced truncation based methods. In the former case, the system is projected onto a subspace to match dominant moments while in the latter case the system is projected onto a subspace both easily controllable and easily observable.

Moment-matching based approaches have been a great success in the electronic design automation community. Those methods do implicit moment-matching in a projection framework and stability, passivity and structure information inherent to RLC circuits can be preserved easily by exploiting the internal structure of RLC formulation. As for moment-matching based MOR in a first-order formulation, AWE [8] is the pioneering work, which suffers from numerical instability owning to explicit moment-matching. To mitigate this problem, Krylov-subspace based methods were proposed [1], where implicit moment-matching is performed in a projection framework. Furthermore, to ensure the stability of the simulation process, PRIMA [6] was developed based on Arnoldi process. PRIMA exploits the positive semi-definiteness of matrices in MNA formulation so that passivity can be easily preserved via congruency transformation. More recently, SPRIM [2] further exploits the block structure of RLC formulation such that, in addition to passivity, structure information inherent to RLC circuits can be preserved at the same time.

While suitable for reduction of large-scale circuits, moment-matching based techniques do not necessarily generate models as compact as desired. Therefore, another approach, truncated balanced realization (TBR), which has been well developed in the control community [5], has been studied intensively. More recently, algorithms [7] were presented to compute guaranteed passive reduced models of controllable accuracy, which is highly appreciated by posing no constrains on the internal structure of the state-space. However, this generality is less appreciated for RLC reduction, where special internal structure is available to preserve passivity less expensively. Also less appreciated is that structure information inherent to RLC circuits such as symmetry, positive semi-definiteness and sparsity, cannot be preserved.

Another issue is that existing TBR-type techniques available in design automation are first-order based and thus cannot handle RLC circuits formulated as second-order systems. As we know, a linear circuit can be equivalently formulated in the form of a first-order system or a second-order system. In fact, it is better to formulate an RLC circuit as a second-order system. One reason is that all matrices in NA are not only positive semi-definite but also symmetric, which makes it easy to preserve structure information inherent to RLC circuits like reciprocity [9]. Another reason is that while the inductance matrix in MNA is usually very large and dense, the susceptance matrix in NA is diagonally dominant and

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can be sparsified by a simple truncation method without disrupting the positive definiteness [11]. However, in the past several years, while second-order moment-matching based approaches have been successfully developed from ENOR [9] to SAPOR [11], second-order TBR-type methods still remain an open problem.

By fully exploiting the symmetric positive definiteness of the system matrices in NA formulation, a novel technique, SBPOR (Second-order Balanced truncation for Passive Order Reduction), is proposed in this paper, which resolves the issues existing in the pioneering work [4] in the control literature by defining second-order gramians based on a symmetric first-order realization. As a result, a real balance can be achieved via congruency transformation without any accuracy loss. Compared with the first-order balanced truncation approaches, SBPOR is a better choice for RLC reduction: in addition to passivity, structure information inherent to RLC circuits can be preserved simultaneously; instead of two quadratic matrix equations, only one linear matrix equation is needed to be solved.

This paper is organized as follows: In Section 2, we review the standard TBR method. In Section 3, we introduce the existing second-order balanced truncation approach and point out some drawbacks of this method. Our new approach, SBPOR, is described in Section 4. Several experimental results are reported in Section 5 to demonstrate the effectiveness of our proposed method. Section 6 concludes the paper.

2. FIRST-ORDER BALANCED TRUNCATION

Given a system in a standard state-space form (A, B, C),

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \tag{1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $u(t) \in R^m$, $y(t) \in R^p$, the controllable and observable gramians are the unique symmetric positive definite solutions to the Lyapunov equations

$$\begin{array}{l} AW_c + W_cA^T + BB^T = 0 \\ A^TW_o + W_oA + C^TC = 0 \end{array} \tag{2}$$

Since the eigenvalues of the product W_cW_o are invariant under a similarity transformation, we can perform a similarity transformation $(A_b = T^{-1}AT, B_b = T^{-1}B, C_b = CT)$ to diagonalize the product W_cW_o

$$\tilde{W_c}\tilde{W_o} = T^{-1}W_cW_oT = diag({\sigma_1}^2, {\sigma_2}^2, \ldots, {\sigma_n}^2) \tag{3}$$

The Hankel singular values of the system, σ_i, are the square roots of the eigenvalues of the product W_cW_o. After the transformation, $\tilde{W}_c$ and $\tilde{W}_o$ are equal and diagonal and such a state-space form (A_b, B_b, C_b) is called balanced

$$\tilde{W}_c = \tilde{W}_o = \Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{4}$$

The Hankel singular values as well as the eigenvalues of gramian product, characterize the *importance* of state variables. States of the balanced system corresponding to the small Hankel singular values are difficult to reach and to observe at the same time. Such states are less involved in the energy transfer from inputs to outputs. Therefore, a general idea of balanced truncation is to transform the system into a balanced form (A_b, B_b, C_b) and to truncate the states that correspond to the small Hankel singular values. We may partition Σ into

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{5}$$

and conformally partition the transformed matrices as

$$A_b = \begin{bmatrix} A_{b11} & A_{b12} \\ A_{b21} & A_{b22} \end{bmatrix} \quad B_b = \begin{bmatrix} B_{b1} \\ B_{b2} \end{bmatrix} \quad C_b = \begin{bmatrix} C_{b1} & C_{b2} \end{bmatrix} \tag{6}$$

The reduced model of order r is obtained by taking the $r\times r$, $r\times m$, $p\times r$ leading blocks of A_b, B_b, C_b, respectively. This truncation leads to a balanced reduced-order system $(A_{b11}, B_{b1}, C_{b1})$. Standard TBR method cannot be relied to preserve passivity. To mitigate this problem, a positive-real TBR (PR-TBR) [7] was proposed to generate the passive models by solving the Lur'e equations, which are quadratic matrix equations and thus more expensive than linear matrix equations like Lyapunov equations.

3. SECOND-ORDER BALANCED TRUNCATION

Given a system in a standard second-order form,

$$\begin{array}{l} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = B_2u(t) \\ y(t) = Pq(t) + Q\dot{q}(t) \end{array} \tag{7}$$

where $u(t) \in R^m$, $y(t) \in R^p$, $q(t) \in R^n$, second-order gramians in [4] are defined based on a first-order realization (1) with 2n-dimensional state $x^T = [q^T \dot{q}^T]$, where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1}B_2 \end{bmatrix}, \quad C = \begin{bmatrix} P & Q \end{bmatrix} \tag{8}$$

The first-order realization (8) has the same input-output behavior as the second-order system (7). In order to directly define gramians for second-order systems, we recall the definition of gramians for first-order systems: the controllability gramian W_c arises from the optimization problem

$$\begin{array}{l} min_{u\in L_2[-\infty,0]} \left(J = \int_{-\infty}^{0} u^2(t)dt \right) \\ \text{subject to} \\ \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \end{array} \tag{9}$$

and the optimum for this problem is $x_0^TW_c^{-1}x_0$, which is the minimum energy for the system to reach a specific state x_0 over all past inputs. Similarly, one of the analogous optimization problems associated with the second-order form can be formed as follows,

$$\begin{array}{l} min_{\dot{q}(0)\in R^n, u\in L_2[-\infty,0]} \left(J = \int_{-\infty}^{0} u^2(t)dt \right) \\ \text{subject to} \\ M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = B_2u(t) \\ q(0) = q_0 \end{array} \tag{10}$$

which minimizes the necessary energy to reach the given q_0 over all past inputs and initial $\dot{q}$. If we compatibly partition the controllability gramian of the first-order realization (8) as $W_c = \begin{bmatrix} R & S \\ S^T & T \end{bmatrix}$, then the optimum for the problem (10) is $q_0^TR^{-1}q_0$, and thus the controllability gramian of the second-order system W_{2c} can be defined as R. Similarly, if we compatibly partition the observability gramian of the first-order realization (8), we can obtain W_{2o}, the observability gramian of the second-order system. The second-order gramians are symmetric positive definite and the eigenvalues of the gramian product $W_{2c}W_{2o}$ are invariant under a similarity transformation

$$\tilde{W}_{2c}\tilde{W}_{2o} = T^{-1}W_{2c}W_{2o}T = diag({\sigma_{21}}^2, {\sigma_{22}}^2, \ldots, {\sigma_{2n}}^2) \tag{11}$$

However, the transformed second-order system in [4] is

$$\begin{array}{l} M_b\ddot{q_b}(t) + D_b\dot{q_b}(t) + K_bq_b(t) = B_{2b}u(t) \\ y(t) = P_bq_b(t) + Q_b\dot{q_b}(t) \end{array} \tag{12}$$

where $M_b = T^TMT, D_b = T^TDT, K_b = T^TKT, B_{2b} = T^TB_2, P_b = PT, Q_b = QT$, which is not really balanced because T^T instead of T^{-1} is used to preserve the symmetry and thus stability of the original system. Unfortunately, the second-order gramians in this coordinate system are not diagonalized. As a result, there is no telling what states are important in energy sense and the accuracy is sacrificed.

4. SBPOR

RLC circuits can be formulated as NA formulation, which is in a second-order form (7) with $M = C$, $D = G$, $K = \Gamma$, $P = 0$, $Q = B_2^T$,

$$\begin{aligned} C\ddot{q}(t) + G\dot{q}(t) + \Gamma q(t) &= B_2 u(t) \\ y(t) &= B_2^T \dot{q}(t) \end{aligned} \tag{13}$$

where $u(t)$, $y(t) \in R^m$ are input currents and output voltages; $\dot{q(t)} \in R^n$ are nodal voltages; G, C, Γ are matrices of conductance, capacitance and susceptance and $C = C^T \geq 0, G = G^T \geq 0, \Gamma = \Gamma^T \geq 0$.

In this paper, instead of using the first-order realization (8), we choose a symmetric first-order realization in descriptor form [3] with 2n-dimensional state $x^T = [q^T \dot{q}^T]$,

$$\begin{aligned} E_d\dot{x}(t) &= A_d x(t) + B_d u(t) \\ y(t) &= B_d^T x(t) \end{aligned} \tag{14}$$

where $E_d = \begin{bmatrix} -\Gamma & 0 \\ 0 & C \end{bmatrix}$, $A_d = \begin{bmatrix} 0 & -\Gamma \\ -\Gamma & -G \end{bmatrix}$, $B_d = \begin{bmatrix} 0 \\ B_d \end{bmatrix}$. Controllability and observability gramians in descriptor form can be computed from the generalized Lyapunov equations [10]. Since C, G, Γ are all symmetric, it follows that $E_d = E_d^T, A_d = A_d^T$ and it is easy to see that, in this symmetrized case, both gramians are equal and are obtained by solving

$$E_d W_{dc} A_d^T + A_d W_{dc} E_d^T + B_d B_d^T = 0 \tag{15}$$

If we compatibly partition the gramians as $W_{dc} = W_{do} = \begin{bmatrix} R & S \\ S^T & T \end{bmatrix}$, then the second-order gramians based on this symmetric realization are $W_{d2c} = W_{d2o} = R$. The gramians are symmetric positive definite and thus the product is symmetric, $(W_{d2c}W_{d2o})^T = W_{d2c}W_{d2o}$. This means the product is orthogonally diagonalizable, i.e. there exists an orthogonal matrix T ($T^{-1} = T^T$) such that

$$\tilde{W}_{d2c}\tilde{W}_{d2o} = T^T W_{d2c} W_{d2o} T = diag({\sigma_{d21}}^2, {\sigma_{d22}}^2, \ldots, {\sigma_{d2n}}^2) \tag{16}$$

So the system can be balanced as

$$\begin{aligned} C_b\ddot{q}_b(t) + G_b\dot{q}_b(t) + \Gamma_b q_b(t) &= B_{2b} u(t) \\ y &= B_{2b}^T \dot{q}_b(t) \end{aligned} \tag{17}$$

where $C_b = T^T C T, G_b = T^T G T, \Gamma_b = T^T \Gamma T, B_{2b} = T^T B_2$. This kind of transformation is known as congruency transformation, which preserves symmetry and definiteness of matrices such that $C_b = C_b^T \geq 0, G_b = G_b^T \geq 0, \Gamma_b = \Gamma_b^T \geq 0$. In the balanced coordinate, we have

$$\tilde{W}_{d2c} = \tilde{W}_{d2o} = \Sigma = diag(\sigma_{d21}, \sigma_{d22}, \ldots, \sigma_{d2n}) \tag{18}$$

We may select the reduced order r and partition Σ into

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{19}$$

and conformally partition the transformed matrices as

$$\begin{aligned} C_b &= \begin{bmatrix} C_r & C_{b12} \\ C_{b21} & C_{b22} \end{bmatrix} \quad G_b = \begin{bmatrix} G_r & G_{b12} \\ G_{b21} & G_{b22} \end{bmatrix} \\ \Gamma_b &= \begin{bmatrix} \Gamma_r & \Gamma_{b12} \\ \Gamma_{b21} & \Gamma_{b22} \end{bmatrix} \quad B_b = \begin{bmatrix} B_r \\ B_{b2} \end{bmatrix} \end{aligned} \tag{20}$$

After truncation, a balanced reduced-order system is given as

$$\begin{aligned} C_r\ddot{q}_r(t) + G_r\dot{q}_r(t) + \Gamma_r q_r(t) &= B_{2r} u(t) \\ y &= B_{2r}^T \dot{q}_r(t) \end{aligned} \tag{21}$$

where $C_r = C_r^T \geq 0, G_r = G_r^T \geq 0, \Gamma_r = \Gamma_r^T \geq 0$, which means the reduced-order system has guaranteed stability, passivity, and reciprocity [9]. Compared with existing techniques, SBPOR shows, for the first time, that a second-order system for RLC circuits can be really balanced via congruency transformation. In addition, only one Lyapunov-like equation is needed to be solved. The basic algorithm flow for SBPOR is given in Fig. 1.

ALGORITHM: SBPOR

Input: C, G, Γ, B

1. Form the symmetric first order realization in descriptor form(14)
2. Solve $E_d W_d^T A_d^T + A_d W_d E_d^T + B_d B_d^T = 0$ for W_d
3. Partition W_d as $\begin{bmatrix} R & S \\ S^T & T \end{bmatrix}$
4. Compute Cholesky factors $W_{d2c} = R = LL^T$
5. Compute SVD of Cholesky product $U\Sigma V = L^T L$
6. Compute the balancing transformation $T = LV\Sigma^{-1/2}$
7. Form the balanced realization as $C_b = T^T C T, G_b = T^T G T, \Gamma_b = T^T \Gamma T, B_{2b} = T^T B_2$
8. Select reduced order r and partition C_b, G_b, Γ_b, B_b conformally
9. Truncate C_b, G_b, Γ_b, B_b to form the reduced realization C_r, G_r, Γ_r, B_r

Output: C_r, G_r, Γ_r, B_r

Figure 1: The algorithm for SBPOR.

5. EXPERIMENTAL RESULTS

In this section, we show examples that illustrate the effectiveness of proposed SBPOR method and compare it with existing MOR approaches.

5.1 Comparison with first-order TBR

Given a circuit in the form (13), we compare the reduced first-order model of dimension $2q$ computed by the standard TBR applied to the equivalent first-order realization (8) and the reduced second-order model of dimension q computed by SBPOR applied to (13) directly. We choose a small circuit as an example so that both impedances and real parts can be compared at all possible reduced orders. The RLC circuit has 4 nodal voltages and thus has a dimension of 4 in a second-order formulation. The equivalent first-order realization has a dimension of 8. As shown in Fig. 2(a),(b),(c), SBPOR outperforms standard TBR at each reduced order. This can be explained from the 'energy' distribution of singular values as shown in Fig. 2(d), where the second-order singular values decay much faster than the first-order ones. The passivity of reduced models can be tested from the real parts. As expected, SBPOR can guarantee the passivity of reduced models while standard TBR cannot. As shown in Fig. 2(a),(b),(c), only in Fig. 2(c), the real part of TBR reduced model is positive at all frequencies and thus the reduced model is passive. Note that standard TBR applied to the equivalent first-order realization(8) also results in a first-order reduced model and thus is not a second-order MOR approach available. Since standard TBR in a first-order form is classic and usually more accurate than PR-TBR [7], we just use it as a criterion to show the accuracy of our new approach.

5.2 Comparison with SAPOR

In the second example, we want to compare our method with moment-matching based second-order MOR approach SAPOR [11]. The example is a RLC circuit, which has 100

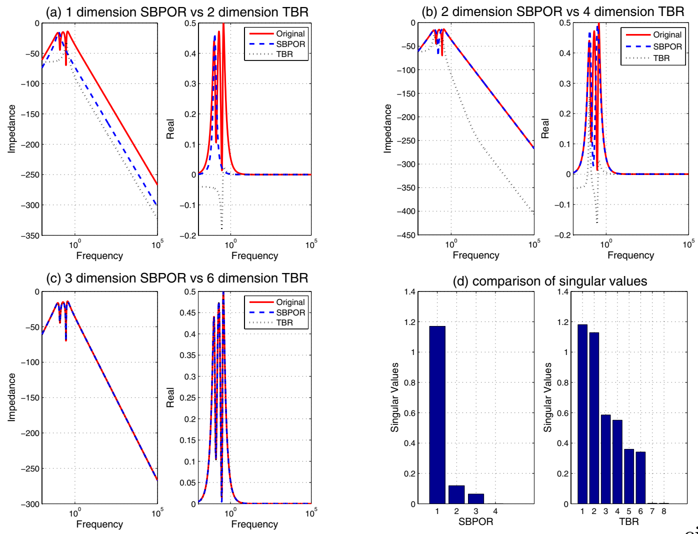

Figure 2: Comparison with first-order TBR.

nodal voltages. The reduced second-order model has a dimension of 2. As shown in Fig. 3(a), SBPOR is globally accurate at all frequencies while SAPOR has very good local behavior around DC but behaves so bad at other frequencies. The error is shown in Fig. 3(b), where the maximum absolute error for SBPOR is about 10 but for SAPOR is almost 100.

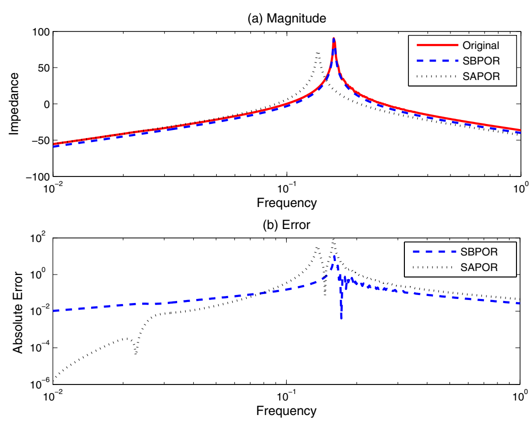

Figure 3: Comparison with SAPOR.

5.3 Comparison with existing second-order TBR

In this part, we want to compare the new method, SBPOR, with existing technique [4] in the control literature, which we name TBR2. The example is a RLC circuit with 100 nodal voltages and the reduced dimension is 10. In Fig. 4(a), we can see that SBPOR outperforms TBR2 obviously. As shown in Fig. 4(b), the maximum absolute error for SBPOR is smaller than 10 while it is almost 100 for TBR2. The reason is that the system in TBR2 is not really balanced and thus the accuracy is sacrificed.

6. CONCLUSIONS

In this paper, we have proposed a novel technique, SBPOR (Second-order Balanced truncation for Passive Order Reduction), for the model order reduction of RLC circuits. To the best of our knowledge, this is the first second-order TBR-type method proposed for passive reduction of RLC

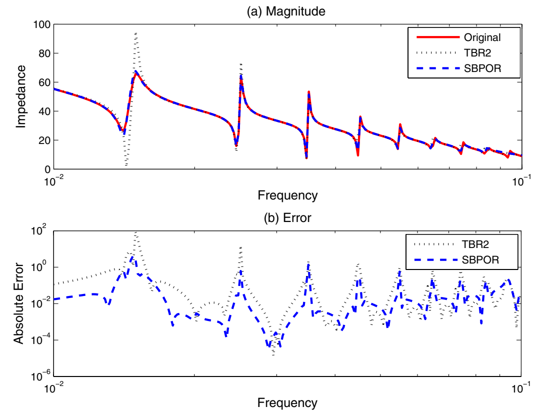

Figure 4: Comparison with pioneering work.

circuits. By fully utilizing the symmetric positive definiteness of the system matrices, new second-order gramians are defined based on a symmetric first-order realization in descriptor form so that the second-order system can be really balanced via congruency transformation without any accuracy loss. The new method mitigated the problems existing in the previous second-order technique [4] in the control community. Compared with existing first-order TBR-type methods [7], the new approach is more suitable for RLC reduction: in addition to passivity, structure information inherent to RLC circuits can be preserved; instead of two quadratic matrix equations, only one linear matrix equation is needed to be solved. Experimental results show that SBPOR is more accurate than existing second-order balanced truncation technique [4] and second-order moment-matching based method SAPOR [11] for the same sizes of reduced models.

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